HL Paper 3

The function *f* is defined on the domain $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ by $f(x) = \ln(1 + \sin x)$.

a. Show that $f''(x) = -\frac{1}{(1+\sin x)}$. [4]

b. (i) Find the Maclaurin series for $f(x)$ up to and including the term in x^4 . [7]

(ii) Explain briefly why your result shows that *f* is neither an even function nor an odd function.

c. Determine the value of $\lim\limits_{x \to 0} \frac{\ln(1+\sin x) - x}{x^2}$. [3]

The integral I_n is defined by $I_n = \int_{n\pi}^{(n+1)\pi} \mathrm{e}^{-x} |\sin x| \mathrm{d}x, \text{ for } n \in \mathbb{N}$.

a. Show that $I_0 = \frac{1}{2}(1 + \mathrm{e}^{-\pi})$. [6]

b. By letting $y = x - n\pi$, show that $I_n = \mathrm{e}^{-n\pi} I_0$. [4]

c. Hence determine the exact value of $\int_0^{\infty} \mathrm{e}^{-x} |\sin x| \mathrm{d}x$. [5]

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a. Given that $y = \ln\left(\frac{1+\mathrm{e}^{-x}}{2}\right)$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^{-y}}{2} - 1$. [5]

b. Hence, by repeated differentiation of the above differential equation, find the Maclaurin series for *y* as far as the term in x^3, showing that two of the terms are zero. [11]

The function f is defined by $f(x) = \mathrm{e}^{-x} \cos x + x - 1$.

By finding a suitable number of derivatives of f, determine the first non-zero term in its Maclaurin series.

Let $f(x) = \mathrm{e}^x \sin x$.

a. Show that $f''(x) = 2\left(f'(x) - f(x)\right)$. [4]

b. By further differentiation of the result in part (a) , find the Maclaurin expansion of $f(x)$, as far as the term in x^5. [6]

Let $f(x) = 2x + |x|$, $x \in \mathbb{R}$.

a. Prove that f is continuous but not differentiable at the point $(0, 0)$. [7]

b. Determine the value of $\int_{-a}^{a} f(x)\mathrm{d}x$ where $a > 0$. [3]

The curves $y = f(x)$ and $y = g(x)$ both pass through the point $(1,\ 0)$ and are defined by the differential equations $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y^2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x^2$ respectively.

a. Show that the tangent to the curve $y = f(x)$ at the point $(1,\ 0)$ is normal to the curve $y = g(x)$ at the point $(1,\ 0)$. [2]

b. Find $g(x)$. [6]

c. Use Euler's method with steps of 0.2 to estimate $f(2)$ to 5 decimal places. [5]

d. Explain why $y = f(x)$ cannot cross the isocline $x - y^2 = 0$, for $x > 1$. [3]

e. (i) Sketch the isoclines $x - y^2 = -2,\ 0,\ 1$. [4]

(ii) On the same set of axes, sketch the graph of f.

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a. Consider the functions $f(x) = (\ln x)^2,\ x > 1$ and $g(x) = \ln(f(x)),\ x > 1$. [5]

(i) Find $f'(x)$.
(ii) Find $g'(x)$.
(iii) Hence, show that $g(x)$ is increasing on $]1,\ \infty[$.

b. Consider the differential equation [12]

$$(\ln x)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = \frac{2x-1}{(\ln x)},\ x > 1.$$

(i) Find the general solution of the differential equation in the form $y = h(x)$.

(ii) Show that the particular solution passing through the point with coordinates $(\mathrm{e},\ \mathrm{e}^2)$ is given by $y = \frac{x^2 - x + \mathrm{e}}{(\ln x)^2}$.

(iii) Sketch the graph of your solution for $x > 1$, clearly indicating any asymptotes and any maximum or minimum points.